Implementing Reflection in Nuprl

Eli Barzilay

eli@barzilay.org

Cornell University

Importance of thesis topic: Reflection

Reflection is a deep idea in Logic, Computer Science, Natural Language and more

(Gödel incompleteness, paradoxes, key idea in Lisp/Scheme)

Reflection is *currently* interesting (PoplMark, PL)

Specifically, reflection is *practical* in a theorem prover:

- allows formalizing meta-mathematical proofs
- can be used to speed proofs, extend the system with verified tactics, meta-proof tools (ACHA90, Knoblock)
- Reasoning about syntax, including programming languages (as a by-product of syntax representation) (PoplMark)

Has been trying for years to achieve reflection in Nuprl.

Requirements for Practical Reflection

We want to implement reflection in Nuprl in a way that is:

- efficient
- elegant
- robust
- *practical*

Thesis results and Impact

- *Direct reflection* of Nuprl's syntax, follows requirements
- Setting a clear approach for direct reflection
- Carried common PL methodology to a theorem prover
- Efficient H.O. representation of Nuprl terms (BTW: HOAS)
- A practical quoting scheme: operator shifting
- Implemented on top of CTT, no changes to the system, including CTT's open-endedness syntax
- Independent of the type theory (mostly)
- Provides a convenient user-interface

Reflection. . .

Always involves an *object* and a *meta language*

In the context of formal languages we need to extend the semantic domain to include syntactic objects:

- ...must allow syntax that denotes (by its semantics) its own syntactic constructs (quotations)
- ...must have functionality for these constructs

  (PL, TP, are dynamic systems)

To make this a reflection:

must connect meta level and the object level

Thesis results and Impact

- *Direct reflection* of Nuprl's syntax, follows requirements
- Setting a clear approach for direct reflection
- Carried common PL methodology to a theorem prover
- Efficient H.O. representation of Nuprl terms (BTW: HOAS)
- A practical quoting scheme: operator shifting
- Implemented on top of CTT, no changes to the system, including CTT's open-endedness syntax
- Independent of the type theory (mostly)
- Provides a convenient user-interface

My Approach

Direct reflection vs. Indirect reflection

- Direct reflection: expose the actual implementation to the user-level.
  (*e.g.*, Scheme/Lisp)
- Indirect reflection: re-implement functionality in your system.
  (*e.g.*, a meta-circular evaluator)

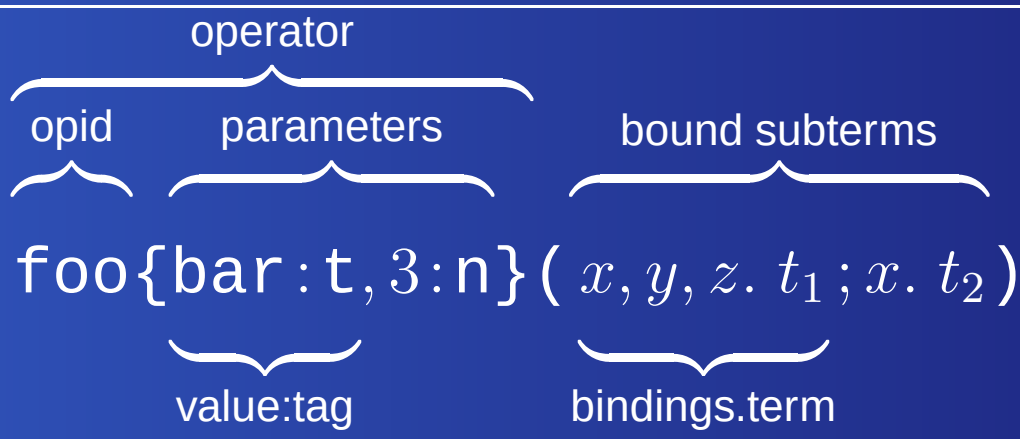
Direct reflection: the obvious choice in PL, not so in logic (various reasons)

Thesis results and Impact

- *Direct reflection* of Nuprl's syntax, follows requirements
- Setting a clear approach for direct reflection
- Carried common PL methodology to a theorem prover
- Efficient H.O. representation of Nuprl terms (BTW: HOAS)
- A practical quoting scheme: operator shifting
- Implemented on top of CTT, no changes to the system, including CTT's open-endedness syntax
- Independent of the type theory (mostly)
- Provides a convenient user-interface

Our working environment: Nuprl Terms

Nuprl terms have uniform structure:



- . . . or *opid*{*v*:*f*, …$_k$}(*x*, …$_m$. *t*; …$_n$)
- Operators are a discrete, open-ended set
- Explicit binding positions
- Exist only at the meta-level

Bindings

- Bindings are strings, a variable reference is a ‘`variable`’ term with a string parameter value
- Example: ‘λ`x.v` $+$ `1`’ is

```
lambda(x.add(variable{x:v};
             natural_number{1:n}))
```

- Related through the family of variable name parameters (‘v’ tag)
- Sophisticated substitution (consistent with user names)

Interaction: Editing & Display Forms

- Work through a structure editor (complex, expansible)
- Display forms (important for Nuprl work)

  ‘λx,y.y’ vs. ‘lambda{}(x.lambda{}(y.variable{y:v}()))’

  ‘ALL(ℕ; x.x < x + 1)’ vs.

```
all{}(.nat{}();
      x.less_than{}(.variable{x:v}();
                    .add(.variable{x:v}();
                         .natural_number{1:n})))
```

- Different renderings for different users

  (compare Eli’s style with Stuart’s)

Some Key Features of Nuprl

- Substitution
  implementation, no hand waving
- Evaluation (Crary and Constable)
  primitive, abstract, & canonical terms
  (untyped lambda calculus)
- Library
  abstractions, display forms, rules, theorems, ML
  (another large part of the system)
- Interactive proof editor, tactics

$\Rightarrow$ Eventually, want to reflect all (precondition: syntax)

Thesis results and Impact

- *Direct reflection* of Nuprl's syntax, follows requirements
- Setting a clear approach for direct reflection
- Carried common PL methodology to a theorem prover
- Efficient H.O. representation of Nuprl terms (BTW: HOAS)
- A practical quoting scheme: operator shifting
- Implemented on top of CTT, no changes to the system, including CTT's open-endedness syntax
- Independent of the type theory (mostly)
- Provides a convenient user-interface

Advantages of Direct Syntactic Reflection

- Flat representations (strings, Gödel numbers) are bad: no structure
- Quote-like contexts fail (destroy congruence)
- Black-box representations cannot mix descriptions with literals
- Standard Type Definitions (re-implementation)
  - Naive approach: imitate the system in itself: `make_lambda("x"; make_var("x"))`
  - Indirect reflection $\Rightarrow$
    - Easy to fall in an exponential trap
    - Incomprehensible without preprocessing
    - Requires intense work, equivalence proof, type theory

Shifted Operators: Quasi-Quoting

Quasi-quotes are important (a modern version of Quine quotes)

- Cannot use contexts $\Rightarrow$ our approach: shifted operators (comes from Allen and Aaron)
- To shift an operator we need to recursively add quotedness tags
- Things that are not shifted in this process retain their meaning

Scheme analogy: `(’+ ’x (’* y ’2))`

Thesis results and Impact

- *Direct reflection* of Nuprl's syntax, follows requirements
- Setting a clear approach for direct reflection
- Carried common PL methodology to a theorem prover
- Efficient H.O. representation of Nuprl terms (BTW: HOAS)
- A practical quoting scheme: operator shifting
- Implemented on top of CTT, no changes to the system, in
- Independent of the type theory (mostly)
- Provides a convenient user-interface

Implementing Shifted Operators

- A new integer parameter value for quotedness,
- No parameter $\equiv$ zero quotedness
- The big question:
  What do we do with bindings of shifted operators?
- The Lisp way: quoted variables are symbols
- In Scheme: problems with hygiene macros
- In Nuprl: the only way we could achieve true direct reflection is to keep keep variables in quoted (shifted) terms as variables — keep the same binding structure
  $\Rightarrow$ we get a HOAS representation

Thesis results and Impact

- *Direct reflection* of Nuprl's syntax, follows requirements
- Setting a clear approach for direct reflection
- Carried common PL methodology to a theorem prover
- Efficient H.O. representation of Nuprl terms (BTW: HOAS)
- A practical quoting scheme: operator shifting
- Implemented on top of CTT, no changes to the system, including CTT's open-endedness syntax
- Independent of the type theory (mostly)
- Provides a convenient user-interface

Implementing Shifted Operators

Did we just lose all explicit names?

```
From: Stuart Allen <sfa@CS.Cornell.EDU>
To: eli@CS.Cornell.EDU, rc@CS.Cornell.EDU
Subject: Tarski sketch
Date: Wed, 28 Feb 2001 15:44:59 -0500 (EST)

Here's my sketch of a Tarski result about truth not being
reflected.  We're assuming we have the type of terms and a "reps"
relation between terms.  We assume that if t reps s then t is
closed.

Notation:
 -x-    is a variable
 Not(t) is the term built from term t by the negation-denoting
        operator
 NOT(t) reps Not(r) if t reps r
 sub(v,t,e) is substitution of e for variable v in t
 subx(t,e) is sub(-x-,t,e)
 SUBX(t,s) reps subx(r,p) if t reps r, and s reps p
 q(t) reps t
 Q(t) reps q(r) if t reps r. Thus, Q(q(t)) reps q(t).

 f(t) is NOT( SUBX(q(t),SUBX(-x-,Q(-x-))) )
 s(t) is subx(f(t),q(f(t)))

   Thus, s(t) is NOT( SUBX( q(t), SUBX(q(f(t)),Q(q(f(t)))) ) )
   Thus, s(t) reps Not( subx( t, subx( f(t),q(f(t)) ) ) )
0) Thus, s(t) reps Not(subx(t,s(t)))

Assume L is a language closed under Not(?).

Let FU(T,tr) where T is a property of terms and tr is a term, mean
1)   forall s,r:term. L(subx(tr,s)) if s reps r
   & forall t:term.
2)      T(Not(t)) iff L(t) and not T(t)
3)    & forall s:term.  if s reps t then ( T(subx(tr,s)) iff T(t))

Then there is not T,tr such that FU(T,tr) thus:

4) Assume FU(T,tr)

5) s(tr) reps Not(subx(tr,s(tr)) by (0)

6) L(subx(tr,s(tr)))  by (4,1,5)

7) T(subx(tr,s(tr)) iff T(Not(subx(tr,s(tr))))  by  (4,3,5)

8) T(Not(subx(tr,s(tr)))) iff
   L(subx(tr,s(tr))) & not T(subx(tr,s(tr))) by (4,2)

9) T(Not(subx(tr,s(tr)))) iff not T(subx(tr,s(tr))) by (8,6)

   T(subx(tr,s(tr))) iff not T(subx(tr,s(tr))) by (7,9)

which is false so (4) is false.

s
```

Implementing Shifted Operators

A few technicalities:

- ‘`rquote`’ parameter, ‘`rquote_term`’, ‘`runquote_term`’
- Editor level operations, work on the concrete level
- Heavy hacking for interface issues

Thesis results and Impact

- *Direct reflection* of Nuprl's syntax, follows requirements
- Setting a clear approach for direct reflection
- Carried common PL methodology to a theorem prover
- Efficient H.O. representation of Nuprl terms (BTW: HOAS)
- A practical quoting scheme: operator shifting
- Implemented on top of CTT, no changes to the system, including CTT's open-endedness syntax
- Independent of the type theory (mostly)
- Provides a convenient user-interface

Implementing Shifted Operators

Visualizing quotations — display forms:

Implementing Shifted Operators

Visualizing quotations — display forms:

Implementing Shifted Operators

Visualizing quotations — display forms:

Implementing Shifted Operators

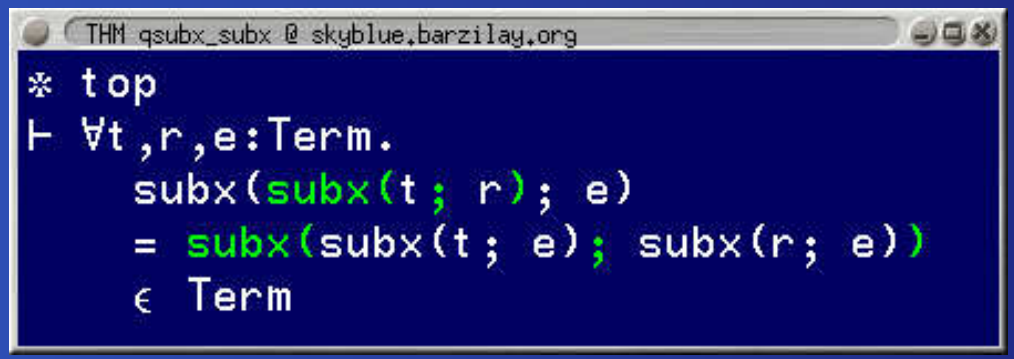
Visualizing quotations — display forms:

Implementing Shifted Operators

Visualizing quotations — display forms:

Implementing Shifted Operators

Visualizing quotations — display forms:



Thesis results and Impact

- *Direct reflection* of Nuprl's syntax, follows requirements
- Setting a clear approach for direct reflection
- Carried common PL methodology to a theorem prover
- Efficient H.O. representation of Nuprl terms (BTW: HOAS)
- A practical quoting scheme: operator shifting
- Implemented on top of CTT, no changes to the system, including CTT's open-endedness syntax
- Independent of the type theory (mostly)
- Provides a convenient user-interface

Semantics of Shifted Terms

- Concrete representation would make '$\underline{\lambda}$' a function of two syntax objects (name+body) which constructs a 'λ' term
- The semantics of our representation uses functions, making it a *higher-order abstract syntax*
- '$\underline{\lambda}$' is a binding operator $\Rightarrow$ takes a function as input
- '$\underline{\lambda}\mathrm{x}.F(\mathrm{x})$' instantiates λ expressions
  $\Rightarrow$ F needs to be a *substitution function*
- $\Rightarrow$ We get the usual benefits
  ...and the usual headaches

Semantics of Shifted Terms

- The semantics we adopt for '$\underline{\lambda}\mathsf{x}\underline{.}F(\mathsf{x})$' is that it denotes the 'λ' term whose body is '$F(u)$' and whose binder is u for some u
  ...the question is which u?
- Using a HOAS: we avoid an answer, denote α-equivalence classes

Use Term for the type that corresponds to α-equivalence classes (CTerm$//\alpha$)
(This work could not be done without Stuart.)

Semantics of Shifted Terms

First intuition: to verify Term-ness you need a shifted operator and check the subterms in a compositional way:

$$\vdash \underline{\mathsf{opid}(v,\ldots.b;\ldots)} \in \mathsf{Term}$$

if $v : \mathsf{Term}, \ldots \vdash b \in \mathsf{Term}$

$\vdots$

Fails with bound variables:

$$\vdash \underline{\lambda}\mathsf{x}\underline{.}\,\mathsf{if}\ \mathsf{x} = \underline{\mathtt{0}}\ \mathsf{then}\ \underline{\mathtt{1}}\ \mathsf{else}\ \underline{\mathtt{2}} \in \mathsf{Term}$$

because $\mathsf{x} : \mathsf{Term} \vdash \mathsf{if}\ \mathsf{x} = \underline{\mathtt{0}}\ \mathsf{then}\ \underline{\mathtt{1}}\ \mathsf{else}\ \underline{\mathtt{2}} \in \mathsf{Term}$

The premise is trivial, but the statement is false

Semantics of Shifted Terms

The problem is *exotic terms*, compare:

- $\underline{\lambda}\mathsf{x}\underline{.}\,\mathsf{if}\ \mathsf{x} = \underline{\mathtt{0}}\ \mathsf{then}\ \underline{\mathtt{1}}\ \mathsf{else}\ \underline{\mathtt{2}}$ (bogus term)
- $\underline{\lambda}\mathsf{x}\underline{.\,\mathsf{if}\ \mathsf{x} = \mathtt{0}\ \mathsf{then}\ \mathtt{1}\ \mathsf{else}\ \mathtt{2}}$ (valid term)
- $\underline{\lambda}\mathsf{x}\underline{.}\,\mathsf{if}\ \mathsf{x} = \underline{\mathtt{0}}\ \mathsf{then}\ \underline{\mathtt{1}}\ \mathsf{else}\ \underline{\mathtt{1}}$ (valid term)
- $\lambda\mathsf{x}.\,\mathsf{if}\ \mathsf{x} = \underline{\mathtt{0}}\ \mathsf{then}\ \underline{\mathtt{1}}\ \mathsf{else}\ \underline{\mathtt{2}}$ (valid function)

Conclusion:

a bound variable can be used only as an argument of a quoted term constructor (syntactic property)

- Addresses the known problem of *exotic terms*
- Difficult to find a formal semantic solution

Semantics of Shifted Terms

- CTerm: given concrete type
- Term $\equiv$ CTerm$//\alpha$ constructed by shifted operators, having the semantics of Term $\rightarrow$ Term *substitution functions e.g.*
  $\underline{\lambda} : \{f : \text{Term} \rightarrow \text{Term} \mid \texttt{is\_subst}_1(f)\} \rightarrow \text{Term}$
- CVar is a subset of CTerm,
  Var $= \{\{x\} \mid x \in \text{CVar}\} \subseteq$ Term
  $\Rightarrow$ *so we do have quoted names*

$\texttt{mkTerm}(o, n, a, f) =$
  $\texttt{mkCTerm}(o, n, a, \lambda i.\ \text{let } \overline{x} = \text{newvar}_{a_i}(f_i) \text{ in}$
    $\texttt{mkBndCTerm}(\overline{x}\lfloor, f_i(\overline{x})\lfloor))\lceil$

Examples:

- $\underline{\lambda}(f) = \texttt{mkTerm}(\text{`}\lambda\text{'}, [\langle 1, f\rangle])$,
- $\underline{\Sigma}(f, g) = \texttt{mkTerm}(\text{`}\Sigma\text{'}, [\langle 0, f\rangle, \langle 1, g\rangle])$

Semantics of Shifted Terms

Defining '`is_subst`':

$$\texttt{is\_subst}_n(f) \equiv$$
$$\exists b : \mathsf{CTerm}.\ \exists \overline{v} : \mathsf{CVar}^n.\ \forall \overline{t} : \mathsf{CTerm}^n.\ f(\overline{t}\uparrow) = b[\overline{t}/\overline{v}]\uparrow$$

$$\texttt{is\_subst}_n(f) \equiv$$
$$\exists b : \mathsf{CTerm}.\ \exists \overline{v} : \mathsf{CVar}^n.\ \forall \overline{r} : \mathsf{Term}^n.\ f(\overline{r}) = b[\overline{r}\downarrow/\overline{v}]\uparrow$$

$$\texttt{is\_subst}_n(f) \equiv$$
$$\exists b_a : \mathsf{Term}.\ \exists \overline{v_a} : \mathsf{Var}^n.\ \forall \overline{t_a} : \mathsf{Term}^n.\ f(\overline{t_a}) = b_a[\![\overline{t_a}/\overline{v_a}]\!]$$

$\Rightarrow$ all equivalent

(Note: based on known concept of concrete substitution)

Semantics of Shifted Terms

...Finally, the '`is_subst`' rule:

- $H \vdash \texttt{is\_subst}(\overline{x}.\ x_i)$ where $1 \leq i \leq \text{len}(\overline{x})$
- $H \vdash \texttt{is\_subst}(\overline{x}.\ \underline{\text{opid}}(\overline{y}.\ b; ...))$

  where $\underline{\text{opid}}$ is some shifted opid

  if $H \vdash \texttt{is\_subst}(\overline{x}, \overline{y}.\ b)$

  $\vdots$

Using it:

- $H \vdash t \in \text{Term}$

  if $H \vdash \texttt{is\_subst}(.\ t)$

$\Rightarrow$ HOAS without heavy type defi nitions

$\Rightarrow$ Convenient rule, easy to mix in descriptions

Using Shifted Terms

- Representation relation, `reps`
- Term induction (actually SubstFunc induction)
- Quotation (‘q’) and unquotation (‘unq’)
- ‘$\uparrow\cdot$’ and ‘$\downarrow\cdot$’

$$\downarrow\uparrow t \ \underline{\texttt{evalsto}} \ t$$

$$\downarrow\underline{\texttt{op(}x\texttt{.}b\texttt{)}} \ \underline{\texttt{evalsto}} \ \texttt{op(}x\texttt{.}\downarrow(b[\uparrow x/x])\texttt{)}$$

$$\uparrow\downarrow t \ \underline{\texttt{evalsto}} \ t$$

$$\uparrow\texttt{op(}x\texttt{.}b\texttt{)} \ \underline{\texttt{evalsto}} \ \underline{\texttt{op(}x\texttt{.}\uparrow(b[\downarrow x/x])\texttt{)}} \quad \text{(canonical ‘op’)}$$

Induction Rule

We prove an induction rule on substitution functions:

$$\forall P : (n : \mathbb{N} \rightarrow \mathsf{SubstFunc}_n \rightarrow Prop).$$
$$(\forall n : \mathbb{N}.$$
$$\forall i : 1...n.\ P_n(\pi_n^i)$$
$$\& \ \forall o : \mathsf{OpId}, k : \mathbb{N}, a : (1...k \rightarrow \mathbb{N}), f : ((i : 1...k) \rightarrow \mathsf{SubstFunc}_{n+a_i}).$$
$$(\forall i : 1...k.\ P_{n+a_i}(f_i)) \Rightarrow P_n(\mathtt{mkSubstFunc}_n(o, k, a, f)))$$
$$\Rightarrow (\forall n : \mathbb{N}, f : \mathsf{SubstFunc}_n.\ P_n(f))$$

The proof is very verbose, and requires a long setup.

Related work: Early Nuprl

Knoblock's thesis: sketch for a reflected Nuprl

- Outline of reflecting Nuprl, focus on the meta-logic ('Metaprl') for user-created tactics
- Naive approach to reflecting syntax (indirect, not open-ended), but identifies many issues that need addressing (antiquotation)

Aitken's thesis, Allen-Constable-Howe-Aitken (ACHA90)

- Design for a reflected logic
- Re-implementation leads to exponential blowup

Related work: HOAS

- Lots of HOAS-related work, became more popular recently (PoplMark, Merlin)
- All face the same fundamental problems: representation, induction, exotic terms, etc
- Usually solved through types, involves heavy math (modal logic, functor categories, set theory)
- Survey in my text

Related work: MetaPRL

- Based on this work
- Main change: use ‘`bterm`’ as the primitive type
- A (top-level) bterm is viewed as a term with free-variable context

Very recent work, continues to change. . . (Merlin’05)

Related work: MetaPRL

- Abstracting more: ‘`mk_term`’, ‘`bnd`’ (HOAS), ‘`vbnd`’, ‘`vsubst`’ (varying arity HOAS)
- Able to express terms in de-Bruijn style: ‘`var`’, ‘`mk_bterm`’
- These result from exposing more: the opid and arity
- More operations: ‘`bdepth`’, ‘`left`’, ‘`right`’, ‘`get_op`’, ‘`subterms`’, ‘`shape`’
- Induction is tricky to define, easy to use
- Aims at abstracting over whole languages and classes of languages

$\Rightarrow$ lost some directedness of the reflected system, got some exponential blowup issues.

Related work: Nominal Logic

Use permutations as the main tool.

- Permutations avoid capture
  example: $(\mathtt{x}\ \mathtt{y}) \bullet (\lambda \mathtt{y}.\mathtt{y}(\mathtt{x}))$
- Easy to deal with (reversible, list concatenation and reverse)
- Works on a very low level (binding positions are treated like the body)
- Variable names are used — but may lose their meaning when permuted

Related work: Nominal Logic

Very recent work by Urban et al builds on nominal logic (Urban-Tasson CADE'05, Urban-Norrish Merlin'05)

- Starts from the same low-level (the 'support' set consists of *all* atoms in a term)
- Introduces *nominal abstractions* that behave like α-equivalence classes —but in definition:
  $a \neq b \wedge x_1 = (a\ b) \bullet x_2 \wedge a\#x_2$
  and in the construction of $\mathbb{A} \rightarrow \Phi$ functions for Λ_α:
  '`if` $b\#t$ `then` $(a\ b) \bullet t$ `else er`'
  —the freshness constraint essentially turns permutation into plain renaming
- The resulting Λ_α setting is similar to our Term (fresh = free-in)

Related work: Nominal Logic

$$\Phi \supseteq \Lambda_\alpha \leftarrow bijection \rightarrow \Lambda_{/\approx} \leftarrow coincide \rightarrow \Lambda_{/=_\alpha}$$

Λ_α ↓ Term

$\Lambda_{/=_\alpha}$ ↓ CTerm//α

Comparison:

- Not derived from simple known syntax principles: elegant, but lots of specific terms (Disagreement Set, Permutation Equality, PSets, Support, Freshness, Fs-PSets, Nominal Abstractions)
- Capture-avoiding substitution does not come for free
- Free induction principle, but not structural
- Currently limited to λ-calculus (recent work on π-calculus)
  $\Rightarrow$ Might work better if used in Nuprl due to uniform binding treatment
- Some HOL/Isabelle problems (functions indirectly via relations)

Related work: Nominal Logic

Major feature of Urban's work: customized induction principle uses context variables for Barendregt-style proofs with the variable convention

- May be possible to slap this on other HOAS inductions in the same way
- If MetaPRL's '`bterm`'s work for this, it will make a more elegant solution

Conclusions

- Syntax can be reflected efficiently and robustly
- Major achievement: direct reflection as a methodology
- Already used — as the foundation for MetaPRL's reflection

Future Work

- Usable induction rule
- Generalized up/down operations
- Formalize syntax-oriented texts
- Enhance general language-oriented reasoning (PoplMark)
- Possible to implement a fully reflected Nuprl, based on Knoblock and ACHA90

Thanks

- Members of my committee: Robert Constable, Greg Morrisett, Barry Perlus
- Stuart Allen who made the formalization possible
- Aleksey Nogin (early ideas, continues in MetaPRL)
- This work was supported by the DoD Multidisciplinary University Research Initiative (MURI) program administered by the Offi ce of Naval Research under Grant #N00014-01-1-0765, and NSF Innovative Programming technology for embedded systems #CCR-0208536